MODEL ANSWERS

AQA

SPECIMEN MATERIAL

Please write clearly, in block capitals.

Centre number ☐☐☐☐☐ Candidate number ☐☐☐☐

Surname ____________________

Forename(s) ____________________

Candidate signature ____________________

A-level MATHEMATICS

Paper 1

Exam Date | Morning | Time allowed: 2 hours

Materials

For this paper you must have:

- The AQA booklet of formulae and statistical tables.
- You may use a graphics calculator.

Instructions

- Use black ink or black ball-point pen. Pencil should be used for drawing.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information

- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice

Unless stated otherwise, you may quote formulae, without proof, from the booklet.
You do not necessarily need to use all the space provided.

Answer **all** questions in the spaces provided.

1 Find the gradient of the line with equation $2x + 5y = 7$

Circle your answer.

[1 mark]

$\frac{2}{5}$ $\quad$ $\frac{5}{2}$ $\quad$ $-\frac{2}{5}$ $\quad$ $-\frac{5}{2}$

$2x + 5y = 7$

$\Rightarrow y = -\frac{2}{5}x + \frac{7}{5}$

2 A curve has equation $y = \frac{2}{\sqrt{x}}$

Find $\frac{dy}{dx}$

Circle your answer.

[1 mark]

$\frac{\sqrt{x}}{3}$ $\quad$ $\frac{1}{x\sqrt{x}}$ $\quad$ $-\frac{1}{x\sqrt{x}}$ $\quad$ $-\frac{1}{2x\sqrt{x}}$

$y = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}\left(2x^{-\frac{3}{2}}\right) = -x^{-\frac{3}{2}} = \frac{-1}{x\sqrt{x}}$

3 When θ is small, find an approximation for $\cos 3\theta + \theta \sin 2\theta$, giving your answer in the form $a + b\theta^2$

[3 marks]

$$\cos 3\theta + \theta \sin 2\theta \approx 1 - \frac{(3\theta)^2}{2} + \theta(2\theta)$$

$$= 1 - \frac{9}{2}\theta^2 + 2\theta^2$$

$$= 1 - \frac{5}{2}\theta^2$$

Turn over for the next question

4 $\mathrm{p}(x) = 2x^3 + 7x^2 + 2x - 3$

4 (a) Use the factor theorem to prove that $x + 3$ is a factor of $\mathrm{p}(x)$

[2 marks]

$p(-3) = 2(-3)^3 + 7(-3)^2 + 2(-3) - 3$

$= -54 + 63 - 6 - 3$

$= 0$

Since $p(-3) = 0$, $x+3$ is a factor of $p(x)$.

4 (b) Simplify the expression $\dfrac{2x^3 + 7x^2 + 2x - 3}{4x^2 - 1}$, $x \neq \pm \dfrac{1}{2}$

[4 marks]

The numerator has a factor of $x+3$ so we can factorise out $x+3$:

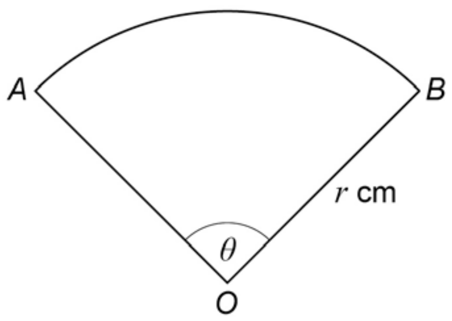
$$\begin{array}{r l}
 & 2x^2 + x - 1 \\
x+3 & \overline{)\,2x^3 + 7x^2 + 2x - 3} \\
 & -(2x^3 + 6x^2) \\
 & \quad 0 \quad x^2 + 2x - 3 \\
 & \qquad -(x^2 + 3x) \\
 & \qquad\quad 0 \quad -x - 3 \\
 & \qquad\qquad -(-x - 3) \\
 & \qquad\qquad\quad 0
\end{array}$$

So, $\dfrac{2x^3 + 7x^2 + 2x - 3}{4x^2 - 1} = \dfrac{(x+3)(2x^2 + x + 1)}{(2x-1)(2x+1)} = \dfrac{(x+3)(2x-1)(x+1)}{(2x+1)(2x-1)}$

$= \dfrac{(x+1)(x+3)}{(2x+1)}$, with $x \neq -\dfrac{1}{2}$.

Turn over for the next question

5 The diagram shows a sector *AOB* of a circle with centre *O* and radius r cm.



The angle *AOB* is θ radians

The sector has area 9 cm^2 and perimeter 15 cm.

5 (a) Show that r satisfies the equation $2r^2 - 15r + 18 = 0$

[4 marks]

Area $= \frac{1}{2}\theta r^2 = 9 \Rightarrow \theta = \frac{18}{r^2}$

Perimeter = arc length + $2r$

$15 = r\theta + 2r$

$15 = r\left(\frac{18}{r^2}\right) + 2r$

$15r = 18 + 2r^2$

$2r^2 - 15r + 18 = 0$

5 (b) Find the value of θ . Explain why it is the only possible value.

[4 marks]

$2r^2 - 15r + 18 = 0$

$(2r-3)(r-6) = 0$

$r = \frac{3}{2}$ or $r = 6$

If $r = 6$, $\theta = \frac{18}{6^2} = \frac{18}{36} = \frac{1}{2}$

If $r = \frac{3}{2}$, $\theta = \frac{18}{(\frac{3}{2})^2} = \frac{18}{(9/4)} = 8$

$8 > 2\pi$ so is larger than a full circle. This cannot be the value of θ, so $\theta = \frac{1}{2}$.

Turn over for the next question

6 Sam goes on a diet. He assumes that his mass, m kg after t days, decreases at a rate that is inversely proportional to the cube root of his mass.

6 (a) Construct a differential equation involving m , t and a positive constant k to model this situation.

[3 marks]

$$\frac{dm}{dt} = -\frac{k}{\sqrt[3]{m}}$$

6 (b) Explain why Sam's assumption may not be appropriate.

[1 mark]

He is assuming his mass will continue to decrease which is not realistic, loss will slow down.

7 Find the values of k for which the equation $(2k-3)x^2 - kx + (k-1) = 0$ has equal roots.

[4 marks]

$(2k-3)x^2 - kx + (k-1) = 0$

To have equal roots, the discriminant $b^2 - 4ac$ must equal 0:

$b^2 - 4ac = k^2 - 4(2k-3)(k-1) = 0$

$k^2 - 4(2k^2 - 5k + 3) = 0$

$7k^2 - 20k + 12 = 0$

$(7k-6)(k-2) = 0$

So, $k = \frac{6}{7}$ or $k = 2$.

Turn over for the next question

8 (a) Given that $u = 2^x$, write down an expression for $\frac{\mathrm{d}u}{\mathrm{d}x}$

[1 mark]

$u = 2^x$

$\frac{du}{dx} = 2^x \ln 2$

8 (b) Find the exact value of $\int_0^1 2^x \sqrt{3 + 2^x}\ \mathrm{d}x$

Fully justify your answer.

[6 marks]

$\int_0^1 2^x \sqrt{3+2^x}\, dx$

Let $u = 2^x$, $\frac{du}{dx} = 2^x \ln 2 \Rightarrow dx = \frac{1}{2^x \ln 2} du$

When $x = 1$, $u = 2$

$x = 0$, $u = 1$.

$\int_1^2 u\sqrt{3+u} \cdot \frac{1}{u \ln 2} du = \int_1^2 (3+u)^{\frac{1}{2}} \frac{1}{\ln 2} du$

$= \frac{1}{\ln 2} \int_1^2 (3+u)^{\frac{1}{2}} du = \frac{1}{\ln 2} \left[\frac{2}{3} (3+u)^{\frac{3}{2}} \right]_1^2$

$= \frac{2}{3\ln 2} \left[(5\sqrt{5}) - (8) \right] = \frac{2}{3 \ln 2} (5\sqrt{5} - 8)$

Turn over for the next question

9 A curve has equation $y = \dfrac{2x+3}{4x^2+7}$

9 (a) (i) Find $\dfrac{dy}{dx}$

[2 marks]

By the quotient rule:

$$\frac{dy}{dx} = \frac{2(4x^2+7) - 8x(2x+3)}{(4x^2+7)^2} = \frac{8x^2 + 14 - 16x^2 - 24x}{(4x^2+7)^2}$$

$$= \frac{2(7 - 12x - 4x^2)}{(4x^2+7)^2}$$

9 (a) (ii) Hence show that y is increasing when $4x^2 + 12x - 7 < 0$

[4 marks]

$(4x^2+7) > 0$ for all x

We also need the numerator to be > 0.

So, $7 - 12x - 4x^2 > 0$

$\Rightarrow 4x^2 + 12x - 7 < 0$

In this case, $\dfrac{dy}{dx} > 0$ and so y is increasing.

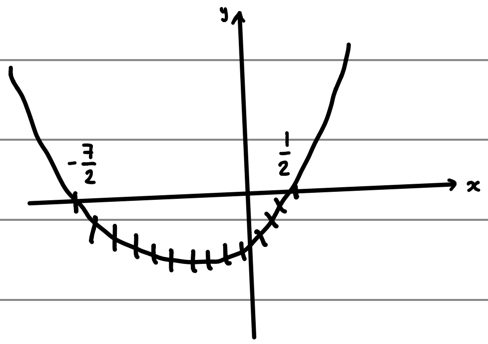
9 (b) Find the values of x for which y is increasing.

[2 marks]

$4x^2 + 12x - 7 < 0$

$(2x-1)(2x+7) < 0$

Critical values: $x = \frac{1}{2}$, $x = -\frac{7}{2}$



$-\frac{7}{2} < x < \frac{1}{2}$

Turn over for the next question

10 The function f is defined by

$$\mathrm{f}(x) = 4 + 3^{-x}, \ x \in \mathbb{R}$$

10 (a) Using set notation, state the range of f

[2 marks]

$(x : x > 4, x \in \mathbb{R})$

10 (b) The inverse of f is f^{-1}

10 (b) (i) Using set notation, state the domain of f^{-1}

[1 mark]

$(x : x > 4, x \in \mathbb{R})$

10 (b) (ii) Find an expression for $\mathrm{f}^{-1}(x)$

[3 marks]

$y = 4 + 3^{-x}$

$x = 4 + 3^{-y}$

$3^{-y} = x - 4$

$-y = \log_3(x-4)$

$y = -\log_3(x-4)$

$f^{-1}(x) = -\log_3(x-4)$

10 (c) The function g is defined by

$$g(x) = 5 - \sqrt{x} \ , \ (x \in \mathbb{R} : x > 0)$$

10 (c) (i) Find an expression for $gf(x)$

[1 mark]

$gf(x) = 5 - \sqrt{4 + 3^{-x}}$

10 (c) (ii) Solve the equation $gf(x) = 2$, giving your answer in an exact form.

[3 marks]

$5 - \sqrt{4 + 3^{-x}} = 2$

$3 = \sqrt{4 + 3^{-x}}$

$9 = 4 + 3^{-x}$

$3^{-x} = 5$

$-x = \log_3 5$

$x = -\log_3 5$

11 A circle with centre C has equation $x^2 + y^2 + 8x - 12y = 12$

11 (a) Find the coordinates of C and the radius of the circle.

[3 marks]

$x^2 + y^2 + 8x - 12y = 12$

$x^2 + 8x + y^2 - 12y = 12$

$(x+4)^2 - 16 + (y-6)^2 - 36 = 12$

$(x+4)^2 + (y-6)^2 = 64$

Centre C: $(-4, 6)$

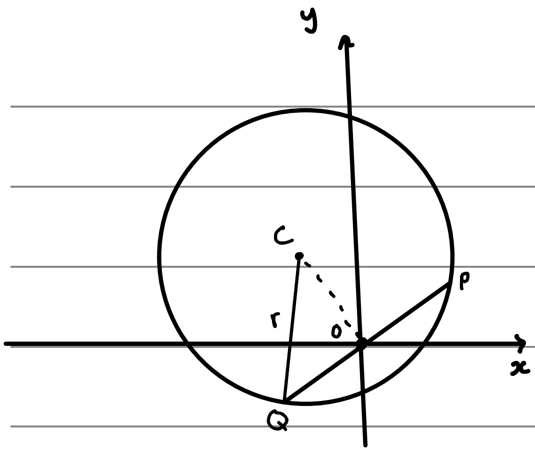
Radius: $\sqrt{64} = 8$

11 (b) The points P and Q lie on the circle.

The origin is the midpoint of the chord PQ.

Show that PQ has length $n\sqrt{3}$, where n is an integer.

[5 marks]



$OC = \sqrt{4^2 + 6^2} = \sqrt{52}$

$CQ^2 = OC^2 + OQ^2$

$r^2 = 52 + OQ^2$

$OQ^2 = 8^2 - 52 = 12$

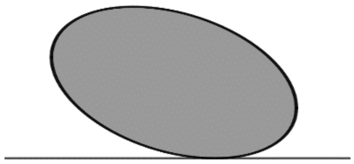
$OQ = \sqrt{12}$

$PQ = 2OQ = 2\sqrt{12} = 4\sqrt{3}$

12 A sculpture formed from a prism is fixed on a horizontal platform, as shown in the diagram.

The shape of the cross-section of the sculpture can be modelled by the equation $x^2 + 2xy + 2y^2 = 10$, where x and y are measured in metres.

The x and y axes are horizontal and vertical respectively.



Find the maximum vertical height above the platform of the sculpture.

[8 marks]

$x^2 + 2xy + 2y^2 = 10$

Implicit differentiation: $2x + 2x\frac{dy}{dx} + 2y + 4y\frac{dy}{dx} = 0$

The minimum and maximum will occur when $\frac{dy}{dx} = 0$:

$2x + 2x(0) + 2y + 4y(0) = 0$

$2y = -2x$

$y = -x$

Substitute $y = -x$ back into the original equation:

$x^2 + 2x(-x) + 2(-x)^2 = 10$

$x^2 - 2x^2 + 2x^2 = 10$

$x^2 = 10$

$x = \pm\sqrt{10}$

Since $y = -x$, $y = \pm\sqrt{10}$.

So the highest point is at $\sqrt{10}$ and the lowest is at $-\sqrt{10}$.

So the distance between these points is :

$$\sqrt{10} - (-\sqrt{10}) = 2\sqrt{10} \text{ m}$$

13 Prove the identity $\cot^2\theta - \cos^2\theta \equiv \cot^2\theta\cos^2\theta$

[3 marks]

$$\text{RHS} = \cot^2\theta\cos^2\theta = \cos^2\theta\,(\operatorname{cosec}^2\theta - 1)$$

$$= \cos^2\theta\left(\frac{1}{\sin^2\theta} - 1\right)$$

$$= \frac{1}{\tan^2\theta} - \cos^2\theta$$

$$= \cot^2\theta - \cos^2\theta \quad = \text{LHS}$$

14 An open-topped fish tank is to be made for an aquarium.

It will have a square horizontal base, rectangular vertical sides and a volume of 60 m^3

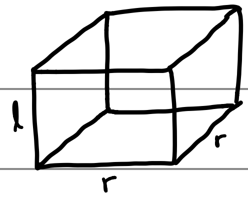
The materials cost:

- £15 per m^2 for the base
- £8 per m^2 for the sides.

14 (a) Modelling the sides and base of the fish tank as laminae, use calculus to find the height of the tank for which the overall cost of the materials has its minimum value.

Fully justify your answer.

[8 marks]



Let r be the sides of the base.

Let l be the height.

Cost: $C = 15r^2 + 8(4rl)$ Volume: $60 = r^2 l$

$C = 15r^2 + 32rl$ $l = \frac{60}{r^2}$

$C = 15r^2 + 32r\left(\frac{60}{r^2}\right)$

$C = 15r^2 + \frac{1920}{r}$

$\frac{dC}{dr} = 30r - \frac{1920}{r^2}$

Cost is minimised when $\frac{dC}{dr} = 0$:

$0 = 30r - \frac{1920}{r^2}$

$\frac{1920}{r^2} = 30r$

$r^3 = \frac{1920}{30} = 64 \Rightarrow r = 4$

Height: $l = \frac{60}{4^2} = 3.75$ m

We need to check that this value of r gives a minimum and not a maximum:

$\frac{d^2C}{dr^2} = 30 + \frac{1920(2)}{r^3}$

At $r = 4$, $\frac{d^2C}{dr^2} = 30 + \frac{1920(2)}{4^3} = 90 > 0$ so it is a minimum.

Therefore, the height should be 3.75m.

14 (b) (i) In reality, the thickness of the base and sides of the tank is 2.5 cm

Briefly explain how you would refine your modelling to take account of the thickness of the sides and base of the tank of the tank.

[1 mark]

To stick the sides together they will need to overlap. Two of the side lengths will need to be $x + 0.5$.

14 (b) (ii) How would your refinement affect your answer to part **(a)**?

[1 mark]

It will cost slightly more as the sides are slightly larger. However, this won't make a significant difference.

15 The height x metres, of a column of water in a fountain display satisfies the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{8\sin 2t}{3\sqrt{x}}$, where t is the time in seconds after the display begins.

15 (a) Solve the differential equation, given that initially the column of water has zero height.

Express your answer in the form $x = \mathrm{f}(t)$

[7 marks]

$$\frac{dx}{dt} = \frac{8\sin 2t}{3\sqrt{x}}$$

$$3\sqrt{x}\,\frac{dx}{dt} = 8\sin 2t$$

$$\int 3\sqrt{x}\,dx = \int 8\sin 2t\,dt$$

$$3\left[\frac{2}{3}x^{\frac{3}{2}}\right] = 8\left[-\frac{1}{2}\cos 2t\right] + c$$

$$2x^{\frac{3}{2}} = -4\cos 2t + c$$

At $t=0,\ x=0$: $2(0) = -4\cos(0) + c$

$c = 4$

$$2x^{\frac{3}{2}} = -4\cos 2t + 4$$

$$x^{\frac{3}{2}} = 2 - 2\cos 2t$$

$$x = (2 - 2\cos 2t)^{\frac{2}{3}}$$

15 (b) Find the maximum height of the column of water, giving your answer to the nearest cm.

[1 mark]

The largest $(2 - 2\cos 2t)$ can be is when $\cos 2t = -1$.

Then $2 - 2(-1) = 4$

So, $x = 4^{\frac{2}{3}} = 2.52\text{m} \Rightarrow x = 252\text{ cm}$

16 A student argues that when a rational number is multiplied by an irrational number the result will always be an irrational number.

16 (a) Identify the rational number for which the student's argument is not true.

[1 mark]

0

16 (b) Prove that the student is right for all rational numbers other than the one you have identified in part **(a)**.

[4 marks]

We will prove by contradiction.

Let a be a rational number and b be an irrational number.

So, we can write a in the form $a = \frac{c}{d}$, $c, d \in \mathbb{Z}$, $d \neq 0$.

Assume ab is rational, so let $ab = \frac{x}{y}$, $x, y \in \mathbb{Z}$, $y \neq 0$.

So $\frac{x}{y} = \frac{cb}{d}$

$b = \frac{dx}{cy}$

But b is irrational so this is a contradiction. So ab must be irrational.

17 $\mathrm{f}(x) = \sin x$

Using differentiation from first principles find the exact value of $\mathrm{f}'\left(\frac{\pi}{6}\right)$

Fully justify your answer.

[6 marks]

$$f'\left(\frac{\pi}{6}\right) = \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6}\right)\cos(h) + \cos\left(\frac{\pi}{6}\right)\sin(h) - \sin\left(\frac{\pi}{6}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}\cos(h) + \frac{\sqrt{3}}{2}\sin(h) - \frac{1}{2}}{h}$$

We know the limit of $\frac{\sin(h)}{h}$ and $\frac{\cos(h)-1}{h}$ so we want to try and express it in terms of these:

$$= \lim_{h \to 0} \left(\frac{\frac{1}{2}(\cos(h)-1)}{h} + \frac{\sqrt{3}}{2} \frac{\sin(h)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\frac{1}{2}\left(-2\sin^2\left(\frac{h}{2}\right)\right)}{2 \cdot \frac{h}{2}} + \frac{\sqrt{3}}{2} \frac{\sin(h)}{h} \right)$$

$$= \left(-\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{2} \right) \left(\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right) + \frac{\sqrt{3}}{2} \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= (0 \times 1) + \left(\frac{\sqrt{3}}{2} \times 1 \right)$$

$$= \frac{\sqrt{3}}{2}$$

END OF QUESTIONS

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